B.Sc. part-II (H). paper-III Tangents and normals 21-07-2020

Problems based on the angle of intersection of two cartesian curves

Q.1 Find the condition that the conics $ax^2+by^2=1$ and $a_1x^2+b_1y^2=1$ shall cut orthogonally.

Soln: The equations of the conics are

$$f(x,y) \equiv ax^2+by^2-1=0 \quad \text{———— (I)}$$

and $$F(x,y) \equiv a_1x^2+b_1y^2-1=0 \quad \text{———— (II)}$$

If conics cut orthogonally at (x,y), then $F_x f_x + F_y f_y = 0$

or, $2a_1 x \cdot 2ax + 2b_1 y\, 2by = 0$ or, $aa_1x^2+bb_1y^2=0$ ———— (III)

But the point (x,y) is common to both (I) and (II),

$\therefore$ Subtracting (II) from (I), we get ———— (IV)

$$(a-a_1)x^2+(b-b_1)y^2=0$$

Thus from (III) and (IV), we get

$$\frac{aa_1x^2}{(a-a_1)x^2} = \frac{-bb_1y^2}{-(b-b_1)y^2}$$

$$\Rightarrow \frac{a-a_1}{aa_1} = \frac{b-b_1}{bb_1} \Rightarrow \frac{1}{a_1}-\frac{1}{a} = \frac{1}{b_1}-\frac{1}{b}$$

$\Rightarrow \frac{1}{a}-\frac{1}{b} = \frac{1}{a_1}-\frac{1}{b_1}$, which is the required condition.

Q.2 Find the angle of intersection of the curves

$$x^2-y^2=a^2, \quad x^2+y^2=a^2\sqrt{2}$$

Soln Let us first the point of intersection of the two curves

The given equations of the curves are

$$F(x,y) \equiv x^2-y^2-a^2=0$$
$$f(x,y) \equiv x^2+y^2-a^2\sqrt{2}=0$$

Adding and subtracting, we get

$$x = \pm\frac{a}{\sqrt{2}}(\sqrt{2}+1)^{\frac{1}{2}} \text{ and } y = \pm\frac{a}{\sqrt{2}}(\sqrt{2}-1)^{\frac{1}{2}}$$

Let θ be the angle of intersection of the curves at (x,y)

Then, we know that $\tan\theta = \dfrac{F_x f_y - f_x F_y}{f_x F_x + f_y F_y} = \dfrac{2x \times 2y - 2x \times (-2y)}{2x\times 2x + 2y \times (-2y)}$

$$\Rightarrow \tan\theta = \frac{8xy}{4x^2-4y^2} = \frac{2xy}{x^2-y^2}$$

Substituting the values of x and y, we get

$\tan\theta = 1 \Rightarrow \tan\theta = \tan 45^\circ$
$\Rightarrow \theta = 45^\circ$. Answer.

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Problems based on Cartesian sub-tangent, sub-normal, tangent and normal. 21-07-2020.

Q.3. Show that the ~~Sub-normal~~ Sub-tangent at any point of the curve $x^m y^n = a^{m+n}$ varies as the abscissa.

Soln: The eqn of the curve is $x^m y^n = a^{m+n}$ ——— (1)

Differentiating with respect to x, we get

$$m x^{m-1} y^n + n x^m y^{n-1} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{m}{n} \frac{x^{m-1}}{x^m} \frac{y^n}{y^{n-1}} = -\frac{m}{n} \cdot \frac{y}{x}$$

The subtangent at any point of the curve (1)

$$= \frac{y}{\frac{dy}{dx}} = -\frac{n}{m} x$$

Hence, the subtangent at any point of the given curve $\propto$ the abscissa, x. Proved

Q.4. Show that in the curve $by^2 = (a+x)^3$, the square of the subtangent varies as the subnormal.

Soln: We know, subtangent $= \frac{y}{y_1}$, Subnormal $= yy_1$

Here, $\because by^2 = (a+x)^3$, $2byy_1 = 3(a+x)^2$ $\therefore yy_1 = \frac{3(a+x)^2}{2b}$

And $\frac{1}{y_1} = \frac{2by}{3(a+x)^2} \Rightarrow \frac{y}{y_1} = \frac{2by^2}{3(a+x)^2} = \frac{2(a+x)^{\cancel{3}}}{3\cancel{(a+x)^2}} = \frac{2}{3}(a+x)$

$\therefore$ Subnormal $= \frac{3(a+x)^2}{2b} = \frac{4}{9}(a+x)^2 \cdot \frac{27}{8b} = \frac{27}{8b}(\text{Subtangent})^2$

$\therefore$ Square of the subtangent varies as the subnormal.